Gap function in the finite Bak-Sneppen model

K. Tabelow
Institut für Theoretische Physik, FU-Berlin, 14195 Berlin, Germany
(Received 14 July 2000; published 20 March 2001)

We present several analytical results on the average lifetime of supercritical avalanches and the long-term behavior of the gap function in the finite Bak-Sneppen model.

DOI: 10.1103/PhysRevE.63.047101 PACS number(s): 05.65.+b

I. INTRODUCTION

Since the first observation of self-organized criticality (SOC) by Bak, Tang, and Wiesenfeld [1], i.e., the occurrence of critical behavior without the necessity of fine tuning of an external parameter, there has been vast interest in this field [2]. The reason is that SOC provides the only known mechanism so far to generate spatiotemporal complexity, which is ubiquitous in Nature. SOC has been observed in a number of open dynamical complex systems in the macroscopic world, such as sandpiles, earthquakes, the stock market, and even the system of species on earth undergoing evolution [2]. Recently it has been observed [3] that SOC also exists at the microsopic level of matter—at the level of gluons and quarks—namely, in systems of interacting soft gluons.

Even though there is great interest, a general mathematical formalism for the description of SOC does not yet exist. For this reason self-organized criticality is very often studied in terms of cellular automata, which take the main features of the real system into account. An example is the Bak-Sneppen model [4] for biological evolution, which is defined as follows [4]. On a d-dimensional lattice with length L random numbers (called fitness) uniformly distributed in $(0,1)$ are assigned to each cell. At every time step the global minimum fitness is detected and together with its nearest neighbors replaced by new random numbers from $(0,1)$. The all-time maximum G of the minimum fitness increases with time. It is called the gap [5]. When the gap jumps to a larger value, *all* random numbers are uniformly distributed above this gap [5,6]. These points in time are separated by local activity called avalanches, where some of the random numbers are smaller than the gap. In the thermodynamic limit ($L\to\infty$), the gap increases up to a critical value f_c where the distribution of the avalanches exhibits a power-law behavior indicating the occurrence of self-organized criticality in the system.

In an attempt to understand the self-organization process, it has been shown [5] that the evolution of the system is governed by avalanche dynamics and that this can be described in terms of an exact equation, namely, the gap equation, which tries to reflect the approach to the self-organized critical state [5]. It has been pointed out in particular that as the gap $G(t)$ approaches the critical value f_c with ongoing time t the system reaches a stationary state with $dG(t)/dt=0$ (in the limit of infinite size L). At this point we would like to stress two facts. (1) For infinite systems the gap function is always a constant in time, because in order to increase the gap *all* (infinitely many) random numbers have to rise above that larger value, for which an infinitely long time interval is needed. (2) For the sake of simulations of the model and its gap function, one has to deal with finite system sizes L. In doing so, we can readily observe that the gap function $G(t)$ exceeds the critical value f_c after a sufficiently large time rather than approaching a stationary limit.

In this paper we quantify the behavior of the gap function for the finite Bak-Sneppen model, when the gap is larger than f_c, and show in particular that the finite system size limits the applicability of the gap function for describing the self-organization process to times shorter than a characteristic time scale defined by the system size. In Sec. II we calculate the average avalanche lifetime in the case of a finite Bak-Sneppen model and compare this with numerical calculations. In Sec. III we discuss the consequences of the results obtained for the description of the system in terms of the gap function.

II. AVERAGE F_0 AVALANCHE SIZE FOR $F_0>F_c$

In the Bak-Sneppen model for a given value $0<f_0<1$ one defines [5] an f_0 avalanche as the activity between successive points in time when all random numbers are larger than f_0. The lifetime (also called size [5]) of such an avalanche is defined as the number of time steps between those points in time. When the lifetime distribution exhibits power-law behavior, the average lifetime diverges. The average avalanche lifetime is a function of the arbitrary paramter f_0. It has been found [5] that in the thermodynamic limit the divergence is like $(f_c-f_0)^{-\gamma}$ (for $f_0<f_c$), where γ is a critical exponent.

However, for finite systems this fails in the near neighborhood of f_c. This can be understood because larger (on average) avalanches cover the whole system so that avalanches that would have lasted much longer in a larger system, are now finished much more quickly. As a consequence the average avalanche size at $f_0=f_c$ does not diverge but takes a finite value. Nevertheless, as f_0 tends to 1, the average diverges. This divergence can be readily calculated by using Eq. (16) from [5]:

$$\frac{d\ln\langle T\rangle_{f_0}}{df_0}=\frac{\langle n_{cov}\rangle_{f_0}}{1-f_0}, \tag{1}$$

where $\langle n_{cov}\rangle_{f_0}$ denotes the average number of cells covered by f_0 avalanches (i.e., the number of cells that changed their fitness at least once during the course of the avalanche) and $\langle T\rangle_{f_0}$ is their average size. This equation is valid for all

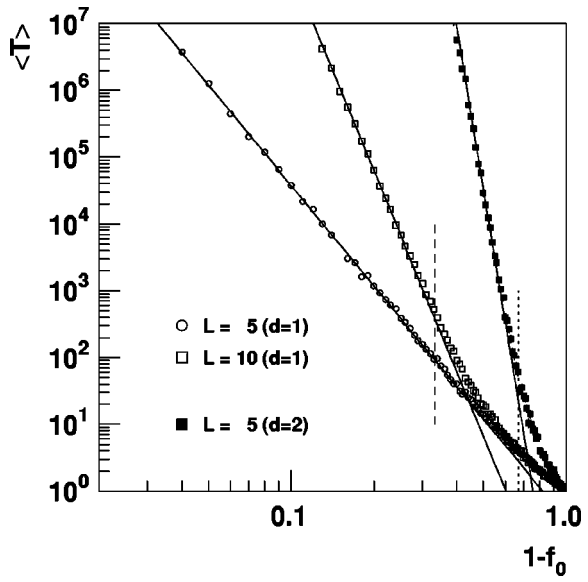

FIG. 1. Average avalanche size as function of $1-f_0$ for different system sizes in the one- (open circles and squares) and two-dimensional (solid squares) Bak-Sneppen model. The dotted and dashed lines indicate the critical values $1-f_c$ in each case. The solid lines correspond to $\sim(1-f_0)^{-L^d}$.

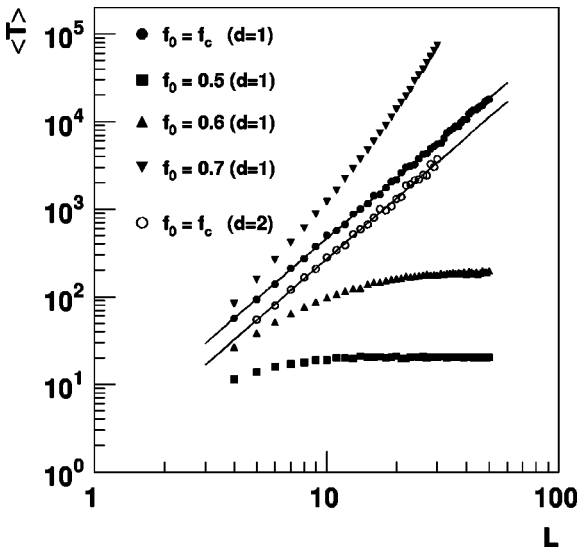

FIG. 2. Average avalanche size as function of L for different f_0 in the one- and two-dimensional Bak-Sneppen model. Only at f_c does full power-law behavior L^E occur.

models [5] of the Bak-Sneppen type independent of their dimension and independent of the value of f_0. Suppose f_0 is sufficiently large, such that $\langle T\rangle_{f_0}$ is large. Then the avalanches cover on average all the cells in the system, i.e.,

$$\langle n_{cov}\rangle_{f_0}=L^d. \tag{2}$$

Inserting this into Eq. (1) and solving this equation leads to

$$\langle T\rangle_{f_0}\sim(1-f_0)^{-L^d}. \tag{3}$$

This means that the average avalanche size diverges at $f_0=1$ for every finite system. This divergence is not of the same interesting dynamical type as that at f_c for the infinite system as can be readily seen. Consider a sufficiently long time interval such that many f_0 avalanches occur. Then on the average at 1 out of $\langle T\rangle_{f_0}$ points in time all of the cells have fitnesses above f_0. If at every time step new random numbers were assigned to all L^d cells in the system, the probability for this to happen would be $(1-f_0)^{L^d}$. Thus we have $1/\langle T\rangle_{f_0}=(1-f_0)^{L^d}$, and hence a behavior as in Eq. (3) aside from the proportionality constant. We interpret this to mean that for f_0 close to 1 only random behavior of the system survives.

We can summarize the dependence of the average avalanche lifetime on f_0 in the finite Bak-Sneppen model as follows. For f_0 smaller then f_c the average $\langle T\rangle_{f_0}$ is proportional to $(f_c-f_0)^{-\gamma}$. For f_0 larger than f_c the dependence is like $(1-f_0)^{-L^d}$. In the near neighborhood of f_c there is a transition between these two regimes. In Fig. 1 the results of numerical calculations for different system sizes and dimensions of the Bak-Sneppen model are shown together with the results of Eq. (3).

It is interesting to consider the average avalanche lifetime at fixed f_0 (especially at f_c) for different system sizes L, because the power-law behavior of the lifetime distributions at f_c is restricted by a cutoff due to the finite system size. Hence, studying the average lifetime at f_c as a function of L yields information about the dependence of this cutoff on the system size. We find through numerical calculations that

$$\langle T\rangle_{f_c}\sim L^E \tag{4}$$

with $E=2.30\pm0.01$ for the one-, $E=2.26\pm0.03$ for the two-, and $E=2.31\pm0.06$ for the three-dimensional Bak-Sneppen model (see Fig. 2). Within the accuracy of the calculation, the value of E is the same in all three dimensions. For the random-neighbor version of this model it was found analytically [7] that the average grows with $L^{0.5}$.

For values of f_0 below f_c the average tends to a finite limit, namely, its value at $L=\infty$. At f_c the full power-law behavior of Eq. (4) is observed, while for larger values of f_0 other functional dependence occurs. Equation (4) can be readily understood if one assumes that the probability distribution $P_{f_c}(T)$ of the avalanche lifetime at f_c exhibits power-law behavior with a cutoff dependent on L:

$$P_{f_c}(T)\sim T^{-\tau}g(TL^{1/\epsilon}), \tag{5}$$

where ϵ is an (model-dependent) exponent describing the cutoff, τ is the exponent of the power law of the lifetime distribution, and $g(x)$ is a scaling function, which decays rapidly when x is large, and tends to a constant for $x\to0$. Calculating the average gives

$$\langle T\rangle_{f_c}\sim L^{(2-\tau)/\epsilon}, \tag{6}$$

which is the observed power-law behavior with $E=(2-\tau)/\epsilon$ (see Fig. 2). $\tau=1.07$ and $\tau=1.245$ [5] for the one- and two-dimensional Bak-Sneppen model, respectively. That is, $\epsilon=0.40$ in one and $\epsilon=0.33$ in two dimensions. It is interesting to see that, although τ *and* ϵ depend on the dimension of the model, $E=(2-\tau)/\epsilon$ seems not to.

III. EVOLUTION OF THE GAP AT LARGE TIMES IN FINITE SYSTEMS

With the help of Eq. (3) we are now able to calculate the long-term behavior of the gap function $G(t)$ using the gap equation [5], which is valid independent of the value of the gap:

$$\frac{dG(t)}{dt}=\frac{1-G(t)}{L^d\langle T\rangle_{G(t)}}. \tag{7}$$

Defining $F(t)=1-G(t)$ and using Eq. (3) this equation reads

$$\frac{dF(t)}{dt}=-\frac{F(t)}{L^d F(t)^{-L^d}}. \tag{8}$$

By separating the variables and integrating this equation, it follows that

$$F(t)=1-G(t)\sim t^{-1/L^d} \tag{9}$$

for sufficiently large t and for every (finite) L. In Fig. 3 a comparison with numerical calculations is shown. It can also be seen (especially for large systems) that before entering this regime (i.e., before the gap has reached f_c) the behavior of the gap function is like $f_c-G(t)\sim t^{-1/(\gamma-1)}$ as calculated in [5,6] from the average lifetime for f_0 avalanches with $f_0<f_c$ by using the gap equation. At f_c the gap function does not reach a stationary value. From the dynamical point of view it is then no longer meaningful to follow its evolution any further. This can be readily understood by recalling that the finite system size L also sets a finite ''system size'' in time, and that considering gap values above f_c needs time scales larger than this.

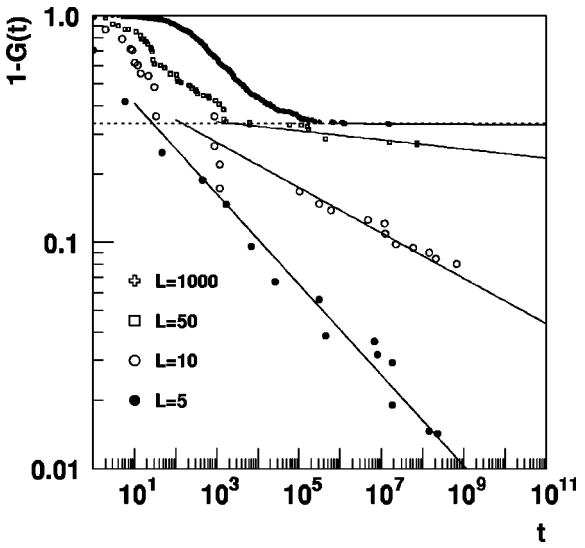

FIG. 3. The difference of the current gap $G(t)$ from 1 as a function of time t for different system sizes in the one-dimensional Bak-Sneppen model. The horizontal dotted line indicates $1-f_c$. The solid lines represent a t^{1/L^d} dependence as described in the text.

IV. CONCLUSIONS

We have shown analytically how the average lifetime depends on f_0 and how the gap function in the Bak-Sneppen evolution model behaves after long time intervals and compared both results successfully to numerical calculations. We observe that the average lifetime at f_c is proportional to L^E, where E is an exponent that is apparently independent of the dimension of the model.

ACKNOWLEDGMENTS

I wish to thank F. J. A. Golcher, T. Meng, and R. Rittel for helpful discussions.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[2] P. Bak, *How Nature Works* (Springer, New York, 1996); H. J. Jensen, *Self-Organized Criticality* (Cambridge University Press, Cambridge, England, 1998), and papers cited therein.

[3] T. Meng, R. Rittel, and Y. Zhang, Phys. Rev. Lett. **82**, 2044 (1999); C. Boros, T. Meng, R. Rittel, K. Tabelow, and Y. Zhang, Phys. Rev. D **61**, 094010 (2000).

[4] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).

[5] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996); and papers cited therein.

[6] M. Paczuski, S. Maslov, and P. Bak, Europhys. Lett. **27**, 97 (1994); T.S. Ray and N. Jan, Phys. Rev. Lett. **72**, 4045 (1994).

[7] J. de Boer, B. Derrida, H. Flyvbjerg, A.D. Jackson, and T. Wettig, Phys. Rev. Lett. **73**, 906 (1994).